

Macroeconomics and Financial Markets

Macroeconomic Dynamics

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Outline

- 1 Macroeconomic dynamics: a simple model
- 2 Solving and Estimating DSGE models
 - Solving for the Recursive Law of Motion

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1 Macroeconomic dynamics: a simple model

2 Solving and Estimating DSGE models

- Solving for the Recursive Law of Motion

The challenge

1 Macroeconomics:

- ▶ Co-movements of key variables over the cycle.
- ▶ Consumption: 1%. Output: 2%. Labor: 2%.
- ▶ Productivity?

2 Finance:

- ▶ Real rate: 1%
- ▶ Equity premium: 6%
- ▶ Equity return volatility: 18%
- ▶ **Sharpe ratio:** 0.3
- ▶ Upward sloping term structure. 20yr - 3mo = 2%.
- ▶ Countercyclical risk premia. Cochrane, pres. address.

A simple DSGE model: part 1

- Source: Uhlig, “Some fiscal calculus”, AER PP, May 2010, 30-34.
- Preferences: (growth-consistent)

$$U = E \left[\sum_{t=0}^{\infty} \beta^t \frac{(c_t \Phi(n_t))^{1-\eta} - 1}{1-\eta} \right]$$

- c_t : cons., x_t : inv., k_t : capital, y_t : output, b_t : debt, q_t : debt price, m_t : net imports, s_t : transfers, τ : tax rates, r_t : cap rental rate, w_t : wage.
- Budget constraint:

$$(1 + \tau^c)c_t + x_t + q_t b_t = b_{t-1} + s_t + m_t + (1 - \tau_t^n)w_t n_t + (r_t - \tau^k(r_t - \delta))k_t \quad (1)$$

- Capital accumulation:

$$k_{t+1} = (1 - \delta)k_t + x_t \quad (2)$$

A simple DSGE model: part 2

- Production:

$$y_t = f\left(\frac{A_t n_t}{k_t}\right) k_t$$

- Factor payments:

$$\begin{aligned} w_t &= A_t f'\left(\frac{A_t n_t}{k_t}\right) \\ r_t k_t &= y_t - w_t n_t \end{aligned}$$

- Productivity: stationary growth. Let's assume iid,

$$\zeta_t = \frac{A_t}{A_{t-1}} = \bar{\zeta} \exp(\epsilon_{A,t}), \quad E_t[\epsilon_{A,t+1}] = 0$$

A simple DSGE model: part 3

- Government budget constraint

$$\begin{aligned} g_t + s_t + b_t &= q_t b_t + \tau_t^n w_t n_t + \tau^c c_t + \tau^k (r_t - \delta) k_t \end{aligned}$$

- Some policy rule. Let's use: (different from paper)

$$\begin{aligned} \tau_t^n - \bar{\tau}^n &= \psi \left((b_t/y_t) - \overline{b/y} \right) \\ g_t/y_t &= \rho_g (g_{t-1}/y_{t-1}) + (1 - \rho_g) \overline{g/y} + \epsilon_{g,t}, \quad E_t[\epsilon_{g,t+1}] = 0 \\ s_t/y_t &= \rho_s (s_{t-1}/y_{t-1}) + (1 - \rho_s) \overline{s/y} + \epsilon_{s,t}, \quad E_t[\epsilon_{s,t+1}] = 0 \end{aligned}$$

- Exogenous net imports:

$$m_t/y_t = \rho_m (m_{t-1}/y_{t-1}) + (1 - \rho_m) \overline{m/y} + \epsilon_{m,t}, \quad E_t[\epsilon_{m,t+1}] = 0$$

- Aggregate feasibility

$$c_t + x_t + g_t = y_t + m_t$$

Equilibrium

Given wages, rental rates, returns, policy rule, evolution of net imports

- Households maximize utility
- Firms maximize profits
- Gov budget constraint is satisfied
- Markets clear

Some parameters, part 1

$\bar{\zeta}$	1.005
δ	0.02
η	2
β	per $\bar{q}^{-1} = \bar{R} = \zeta^\eta / \beta = 1.01$
$\bar{\tau}^n$	0.28
τ^c	0.05,
τ^k	0.36
$\overline{b/y}$	4*0.63
$\overline{g/y}$	0.18
$\overline{m/y}$	0.04
labor share	0.62
$\overline{c/y}$	0.59 (implied)
$\overline{x/y}$	0.27 (implied)
$\overline{s/y}$	0.07 (implied)
ψ	low, with stable dynamics

Source: Uhlig, AER PP, May 2010 and Trabandt-Uhlig, JME (2011).

Some parameters, part 2

abbrev.	expression	value
$\theta =$	$\frac{f'(\bar{n}/\bar{k})\bar{n}/\bar{k}}{f(\bar{n}/\bar{k})} = \bar{w}\bar{n}/\bar{y}$	0.62
$1/\omega_D =$	$\frac{-f''(\bar{n}/\bar{k})\bar{n}/\bar{k}}{f'(\bar{n}/\bar{k})}$	0.38
$\kappa =$	$\frac{-\Phi'(\bar{n})\bar{n}}{\Phi(\bar{n})} = \frac{1-\bar{\tau}^n}{1+\tau^c} \theta \frac{\bar{y}}{\bar{c}}$	0.72
$1/\omega_S =$	$\left(\frac{\Phi''(\bar{n})\bar{n}}{\Phi'(\bar{n})} + \kappa \right)$	0.64
$(= 1/\text{"Frisch elasticity"} - (1 - 1/\eta)\kappa)$		

Second line: per assuming Cobb-Douglas production function.

Forth line: per assuming a Frisch elasticity equal to 1.

Some parameters, part 3

Dynamic policy rule parameters and shock variances: choose or estimate. Units: percent.

$\sigma_A = \sqrt{E[\epsilon_{A,t}^2]}$	0.8
ψ	0.9
ρ_g	0.9
$\sigma_g = \sqrt{E[\epsilon_{g,t}^2]}$	0.2
ρ_s	0.9
$\sigma_s = \sqrt{E[\epsilon_{s,t}^2]}$	0.2
ρ_m	0.9
$\sigma_m = \sqrt{E[\epsilon_{m,t}^2]}$	0.2

Take FONCs. Detrend, log-linearize, analyze

- $\tilde{c}_t = c_t/A_t$, etc..
- \bar{c} : steady state of \tilde{c}_t .
- $\hat{c}_t = \ln(\tilde{c}_t) - \ln(\bar{c})$: log-deviation.
- Some log-linear calculus:

$$\begin{aligned}x_t y_t &= z_t \rightarrow \hat{x}_t + \hat{y}_t = \hat{z}_t \\x_t + y_t &= z_t \rightarrow \bar{x}\hat{x}_t + \bar{y}\hat{y}_t = \bar{z}\hat{z}_t \\x_t &= h(y_t) \rightarrow \hat{x}_t = \frac{h'(\bar{y})\bar{y}}{h(\bar{y})}\hat{y}_t\end{aligned}$$

- More sensible “hats”: (hundredth of) percentage points for the tax rate, $\hat{\tau}_t^n = \tau_t^n - \bar{\tau}^n$. Allow zero per expressing \hat{b}_t , \hat{g}_t , \hat{s}_t and \hat{m}_t relative to steady state output, e.g. $\hat{b}_t = (\tilde{b}_t - \bar{b})/\bar{y}$.
- Analyze: Log-linearized system \rightarrow recursive law of motion \rightarrow impulse responses, policy impacts, etc..

All log-linearized equations, part 1

- Labor market:

$$\begin{aligned}\hat{w}_t &= \frac{1}{\omega_S} \hat{n}_t + \frac{1}{1 - \bar{\tau}^n} \hat{\tau}_t^n + \hat{c}_t \\ \hat{w}_t &= \frac{1}{\omega_D} (\hat{k}_t - \hat{n}_t)\end{aligned}$$

- Production and feasibility:

$$\begin{aligned}\hat{y}_t &= \theta \hat{n}_t + (1 - \theta) \hat{k}_t \\ \hat{y}_t + \hat{m}_t &= \frac{c}{\bar{y}} \hat{c}_t + \frac{x}{\bar{y}} \hat{x}_t + \hat{g}_t\end{aligned}$$

- Technological progress:

$$\hat{\zeta}_t = \epsilon_{A,t}$$

All log-linearized equations, part 2

- Policy rules:

$$\begin{aligned}\hat{\tau}_t^n &= \psi (\hat{b}_t - \hat{y}_t) \\ \hat{g}_t - \hat{y}_t &= \rho_g (\hat{g}_{t-1} - \hat{y}_{t-1}) + \epsilon_{g,t} \\ \hat{s}_t - \hat{y}_t &= \rho_s (\hat{s}_{t-1} - \hat{y}_{t-1}) + \epsilon_{s,t}\end{aligned}$$

- Government budget constraint. Define $\tilde{\delta} = \bar{\zeta} - 1 + \delta$.

$$\begin{aligned}\hat{g}_t + \hat{s}_t + \bar{\zeta}^{-1} \hat{b}_{t-1} - \overline{b/y} \bar{\zeta}^{-1} \hat{\zeta}_t \\ = \overline{b/y} \bar{q} \hat{q}_t + \bar{q} \hat{b}_t + \theta \hat{\tau}_t^n + \theta \bar{\tau}^n (\hat{w}_t + \hat{n}_t) \\ + \tau^c \overline{c/y} \hat{c}_t + \tau^k \left(1 - \theta - \frac{\delta}{\tilde{\delta}} \overline{x/y} \right) \hat{k}_t + \tau^k (1 - \theta) \hat{r}_t\end{aligned}$$

- Net imports:

$$\hat{m}_t - \hat{y}_t = \rho_m (\hat{m}_{t-1} - \hat{y}_{t-1}) + \epsilon_{m,t}$$

All log-linearized equations, part 3

- Capital accumulation and rental rates:

$$\begin{aligned}\hat{k}_t &= (1 - \tilde{\delta})\hat{k}_{t-1} + \tilde{\delta}\hat{x}_{t-1} - \hat{\zeta}_t \\ 0 &= (1 - \theta)\hat{r}_t + \theta\hat{w}_t\end{aligned}$$

- Return on capital:

$$\hat{R}_t^{(k)} = \left(1 - (1 - (1 - \tau^k)\delta)\beta\bar{\zeta}^{-\eta}\right) \hat{r}_t$$

All log-linearized equations, part 4

- Lagrange multiplier λ_t on HH budget constraint:

$$\hat{\lambda}_t = -\eta \hat{c}_t - (1 - \eta) \kappa \hat{n}_t$$

- Asset pricing:

$$1 = E_t \left[\beta \frac{\lambda_{t+1}}{\lambda_t} R_{t+1}^{(k)} \right]$$

$$q_t = E_t \left[\beta \frac{\lambda_{t+1}}{\lambda_t} \right]$$

becomes

$$0 = E_t \left[\hat{\lambda}_{t+1} - \hat{\lambda}_t + \hat{R}_{t+1}^{(k)} \right]$$

$$0 = E_t \left[\hat{\lambda}_{t+1} - \hat{\lambda}_t - \hat{q}_t \right]$$

Count equations and unknowns

- Four exogenous disturbances: $\epsilon_{A,t}, \epsilon_{m,t}, \epsilon_{g,t}, \epsilon_{s,t}$
- 16 unknowns:
 - ▶ $\hat{n}_t, \hat{c}_t, \hat{y}_t, \hat{k}_t, \hat{x}_t, \hat{m}_t, \hat{\zeta}_t$
 - ▶ $\hat{w}_t, \hat{r}_t, \hat{q}_t, \hat{R}_t^{(k)}, \hat{\lambda}_t$
 - ▶ $\hat{\tau}_t^{(n)}, \hat{g}_t, \hat{b}_t, \hat{s}_t$
- 15 equations.
- Note: we did not log-linearize the household budget constraint.
- 16 equations? Walras' law!

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The model in general form

$$E_t[Fx_{t+1} + Gx_t + Hx_{t-1}] = D\epsilon_t$$

where

- $x_t \in \mathbb{R}^n$ is a vector of endogenous variables, known at date t .
- $\epsilon_t \in \mathbb{R}^n$ is a vector of exogenous shocks, iid, $E_t[\epsilon_{t+1}] = 0$,
- F, G, H, D are $n \times n$ matrices.

The method of undetermined coefficients

Idea: Solve

$$E_t[Fx_{t+1} + Gx_t + Hx_{t-1}] = D\epsilon_t$$

for the **recursive law of motion** (RLOM):

$$x_t = Px_{t-1} + Q\epsilon_t$$

Treat the matrices P , Q as “**unknown coefficients**”.

Note and disclaimer: the approach and the following calculations require some special conditions, which are often, but not always satisfied.

Solving, step 1: plug in twice

$$\begin{aligned} E_t[Fx_{t+1} + Gx_t + Hx_{t-1}] &= D\epsilon_t \\ x_t &= Px_{t-1} + Q\epsilon_t \end{aligned}$$

- Plug in:

$$(FP + G)x_t + Hx_{t-1} = D\epsilon_t$$

- Plug in again:

$$(FP^2 + GP + H)x_{t-1} + (FP + G)Q\epsilon_t = D\epsilon_t$$

- Compare coefficients:

$$0 = FP^2 + GP + H$$

$$D = (FP + G)Q$$

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- Compare coefficients:

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Solving, step 2: solve for P

- Multiply with F^{-1} . Let $\tilde{G} = F^{-1}G$ and $\tilde{H} = F^{-1}H$. To solve:

$$0 = P^2 + \tilde{G}P + \tilde{H}$$

- Suppose x is an eigenvector of P to eigenvalue λ . Then,

$$0 = (\lambda^2 + \lambda\tilde{G} + \tilde{H})x$$

- Rewrite this in stacked form:

$$\lambda \begin{bmatrix} \lambda x \\ x \end{bmatrix} = \begin{bmatrix} -\tilde{G} & -\tilde{H} \\ I & 0 \end{bmatrix} \begin{bmatrix} \lambda x \\ x \end{bmatrix}$$

- So: solve this eigenvalue problem. Find $2n$ eigenvalues. If exactly n are stable, $(\lambda_i, x_i)_{i=1}^n$, find P per diagonalization:

$$P = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \left(\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \right)^{-1}$$

Solving, step 3: solve for Q

It remains to solve

$$D = (FP + G)Q$$

for Q . With the solution for P ,

$$(FP + G)^{-1}D = Q$$

Uhlig's Toolkit



$$0 = Ax_t + Bx_{t-1} + Cy_t + Dz_t \quad (3)$$

$$0 = E_t[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t] \quad (4)$$

$$z_{t+1} = Nz_t + \epsilon_{t+1}; \quad E_t[\epsilon_{t+1}] = 0, \quad (5)$$

- ▶ x_t , $m \times 1$: endogenous state vector,
- ▶ y_t , $n \times 1$: other endogenous variables,
- ▶ z_t , $k \times 1$: exogenous stochastic processes,
- ▶ $A, B, C, D, F, G, H, J, K, L, M, N$: matrices. C : size $l \times n$, rank n .

● Recursive law of motion:

$$x_t = Px_{t-1} + Qz_t$$

$$y_t = Rx_{t-1} + Sz_t$$

Dynare

- **Dynare**: a Matlab-based program, created by Michel Juillard with a community of scholars. Google-search for "Dynare", follow download and installation instructions.
- “addpath c:\dynare\4.4.0\matlab”
- Given (nonlinear) equations of a DSGE model, Dynare ...
 - ▶ solves for the steady state,
 - ▶ approximates the dynamics around the steady state
 - ▶ — first-order (“log-linearization”)
 - ▶ — higher-order
 - ▶ Simulates
 - ▶ Estimates, using MCMC methods.
- “dynare modelfile.mod”

Introduction to Dynare per example

- Inspiration: Barillas-Colacito-Kitao-Matthes-Sargent-Shin, “Practicing Dynare,” draft, NYU 2007.
- a few corrections, different model plus slight modification for Dynare 4.4.0
- State the model. Pick parameters.
- Solve with Dynare, simulate data with Dynare:
UhligApproximate.mod
- Note: datasaverVersion04.m saves the data to simuldata.m
- Estimate with Dynare, using the simulated data:
UhligEstimate.mod